

Fearless Evolution:
Fearful Symmetry as Principle in Emergence and in Adaptive Leaps in Boolean Landscapes

Abstract: The computational logic of even small Boolean systems generates landscapes of hundreds or thousands of basins of attraction. Adaptive walks on a fitness landscape encounter daunting barriers in the evolution of dynamic systems; classic strategies (random walks, genetic algorithm, etc.) leave the system passing through low fitness areas or jumping to unknown areas of the landscape. The high degree of symmetry in Boolean XOR Rings generates emergent landscapes whose attractors are related through symmetry groups. The transforms that define these symmetry groups, by their nature, are capable of transforming the attractor of one basin in a symmetry group into the attractor of another basin in the same group. Thus the emergence of symmetry groups constitutes a computationally well defined way to move from one attractor to another in a landscape as an alternative to classic strategies. Symmetry is discussed as a self-transcending construction in emergence.

Computational models that generate fitness landscapes are one way for conceptually organizing the time course (evolution) of adaptive dynamical systems in a broad range of content areas including biological evolution (Gravner, Pitman, & Gavrilets, 2007), ecological relations such as predator-prey (Brown, Cohen, & Vincent, 2007), information (Yossi & Poli, 2005) and organizational behavior (Haslett, 2005; Winter, Cattani, & Dorsch, 2007). The landscape metaphor generated by such computational models is expressed in language evoking geographical terrain in two converse forms. We begin with the peak metaphor which is critically defined by high terrain (attractors that have high fitness) surrounded by valleys of less fitness. Later we will return to the converse watershed form of the landscape that speaks in term of basins with attractive wells. In the peak metaphor, across time a system moves (evolves) in a way that increases its fitness until it attains a local peak (attractor). In the general case, a particular peak, while locally high in fitness relative to surrounding valleys, may have lower fitness than other peaks on the landscape. The peak is a local optimum that “traps” an adaptive walk (Kauffman, 1993) because small movements in any direction mean lowering its fitness. A fundamental puzzle arises: How does a system navigate from one peak (attractor) to another attractor (possibly with higher fitness) given that valleys of low fitness act as barriers to movement? Mutation (e.g., Holland 1995) allows a system to leap some distance away from its local peak but with no information about where (in a valley, on a hillside, or on a peak) it will land; the unknown result of this leap is nontrivial because fitness peaks are hard to find (Skellet, Cairnes, Geard, Tonkes, & Wiles; 2005). Crossover recombination tends to become trapped at local optima.

In computational modeling, landscapes emerge from lower-order generating process (e.g., a network of Boolean nodes). Here we focus on how symmetry imbues the emergence of a Boolean landscape and how this “fearful symmetry” (Stewart & Golubitski, 1992) is a basis for making

“fearless” leaps that take the system from one attractor (high fitness) precisely to another attractor. In this paper we will use our development of a Boolean phase portrait to describe a new, symmetry-based navigation strategy by which information contained solely in one attractor both codes the existence of other attractors and codes the kind of perturbations of the system that will provoke it to leap directly to another of these other attractors. This provides a new alternative for solving important puzzles in the evolution of a system.

Computational models that generate adaptive landscapes come in many forms (e.g., Winter, Cattani, & Dorsch, 2007) and the NK Boolean models developed by Kauffman (e.g., 1993) are a quintessential example. In Boolean computational models N nodes connected to K other nodes are the generating processes from which a landscape emerges. In this paper we will use a highly symmetrical node structure to demonstrate that the resulting symmetry in the emergent landscape can be detected by the Boolean Meta-TAO phase portrait developed by Malloy, Butner, Cooper, and Smith, 2007 and, more importantly, that the phase portrait reveals a new method for the system to move among attractors on the landscape.

As noted above, results of computational models are typically couched in a landscape metaphor where attractors are represented by peaks exhibiting high fitness; these peaks are separated by low fitness valleys. Conversely, the landscape can be portrayed as watershed basins each with an attractor (well) where into which water runs (e.g., Kauffman, 1995, p. 254). We will use peaks and basins interchangeably but due to our previous work (e.g., Malloy, Jensen & Song, 2005) our model will be presented in terms of watershed basins that have tributaries flowing like streams down from ridges into attractors at the bottom of valleys. Moreover, in this paper we will *not* focus on the definition of fitness nor any other “potential” that leads the system to flow toward an attractor (a peak in the one

manner of speaking or a well in the other). Rather, assuming that systems do move toward attractors, we focus on two puzzles: How attractors code information about other attractors and how a system moves directly from one attractor to another without passing through the areas of lesser attraction between attractors.

Symmetry

The first stanza of William Blake's poem, *The Tiger*, (Tiger, Tiger, burning bright/ In the forests of the night,/ What immortal hand or eye/ Dare frame thy fearful symmetry?) is evocative of the power of describing the universe in symmetrical terms that has persisted up to modern dynamical thinking (Stewart and Golubitsky, 1992; Zee, (1986). Assuming, particularly in the way Stewart and Golubitsky do, that it is useful to describe a dynamical universe in terms of symmetry and symmetry breaking then, we argue, it is useful to describe the emergence of landscapes in terms of symmetry and symmetry breaking.

Symmetry Groups. Before we describe how symmetry gets broken as a landscape emerges, we will give a general account of symmetry. Symmetry is defined in terms of transformation. Any transformation that leaves an object apparently unchanged is a symmetric transformation (Stewart & Golubitsky, 1992, p. 28). For the transformation to be nontrivial something must have changed but to speak of symmetry the object in some sense must appear the same before and after the transformation. An infinitely long horizontal line can be translated to the right and it will appear identical; it is said to have translational symmetry. An equilateral triangle whose upper vertex points at 12 o'clock can be rotated 120 degrees about its center and appear identical. It has rotational symmetry through 120 degrees. But the triangle does not have rotational symmetry through, for example, twenty-seven degrees. Assuming we have a frame of reference, an equilateral triangle rotated twenty-seven degrees

will appear different before than after a twenty-seven degree rotation. Stewart and Golubitsky (1992) place symmetry within group theory. A group is a closed set of transforms such that when any two transforms from this set are combined they produce the results of another transform in the set. Suppose with our equilateral triangle we define the closed set of six transforms as clockwise or counterclockwise rotations through 0, 120, and 240 degrees. Combining any two of these six transforms produces one of three triangles that are identical in appearance; no matter how many of the transforms are applied, in whatever order, the application of the transforms will result in one of these three triangles. The three triangles, identical in appearance, comprise a symmetry group defined by the six transformations. (The vertices of the triangles in this symmetry group point at 12, 4, and 8 o'clock). But there are other symmetry groups (of three triangles each). For example, if we start with a new equilateral triangle whose upper vertex points at 1 o'clock and use the same six transforms we would get another group of three identical equilateral triangles (whose vertices point at 1, 5 and 9 o'clock). Thus there are an infinite number of symmetry groups of equilateral triangles, each group with three triangles identical in appearance and each group defined by the same six transforms). We will demonstrate symmetry groups in the attractor basin landscape of NK Boolean systems. The consequence of such symmetry groups is that, given that a system is "stuck" in an attractor on a landscape, the transforms that define these symmetry groups constitute a new method for searching for other attractors.

The Extended Curie Principle. Stewart and Golubitsky (1992) summarize and extend the Curie Principle which proposes that there is a meta-symmetry across symmetries found in physical causes and effects. Curie's original principle stated that (1) the symmetries of causes reappear in their effects and (2) that asymmetries found in effects will be reflected in their causes. Stewart and

Golubitsky revise and extend this principle based on modern analysis. In their extended version, the states of physical systems come in sets and these sets of states are related by symmetry. In their words, “a symmetric cause produces one from a set of symmetrically related effects.” They use the example of morning dew drops on a spider web. Think of a web idealized as an infinite horizontal line and think of night time humidity in the air coating it uniformly so there is a perfect column of water around web. At this point the water-web system has perfect translational symmetry; move it right or left and it appears the same. The distance of movement does not matter; if we move it 0.1 cm to the right or 2 cm to the right the result is that, in either case, it appears the same. This symmetrical state is unstable due to the surface tension of water; so the symmetry “breaks” and beads form along the web. The water-web system no longer has perfect translational symmetry but it still has symmetry. The beads, ideally, are spaced equally and each, ideally, is a sphere with the web running through its center. Say, under certain conditions, the beads are equally spaced 2 cm apart; if we translate the water-web system 2 cm right or left it will appear the same; an observer who blinks while we are translating it will see the long necklace of dew drops as the same. But, once the beads form, the amount of translation matters; if we translate it 0.5 cm, the blinking observer will notice a slight shift in the position of the beads. Some translational symmetry has been lost when the surface tension of the water caused beads.

These beads are formed by geometric and symmetrical forces in the molecules of the water, which, ideally, produce spheres separated by an equal distance; the symmetry of the cause (molecular forces) is found in the effects (beads) as Curie proposed. But, and here is Stewart and Golubitsky's extension, the particular string of beads that emerges is only one of an infinite set of possible symmetry groups where each group is offset from the other groups by any arbitrary lateral distance. We could define one such group in terms of an arbitrary coordinate system in which one bead is at 0 another at

+2 cm, another at -2cm, etc. A second group can be defined as having a bead at 0.1 cm, another at +2.1 cm, another bead at -1.9 cm, etc. Physically only one such group would come into existence but potentially there are an infinite number of groups that could have come into the existence. In our idealized example, the beaded water that appears on the web on one particular morning will be slightly offset from the beaded water that forms on the web on another morning. When the symmetry of the perfectly coated web is broken (i.e., there is less symmetry in the beads than in the column of water) the symmetry is not lost but potentially exists in the other symmetry groups that might have formed. Across all the possible beaded web necklaces that can potentially come into existence the symmetry is the same as it was in the unbroken column of water. In Stewart and Golubitsky's perspective, finding one broken symmetry in a system implies the existence of other broken symmetries in that system; that is, finding one particular necklace of dew drops on a web implies the possible existence of other such necklaces. Symmetry is not so much lost as shared across families of symmetry groups.

Emergent Landscapes and the Extended Curie Principle. Our conjecture, based on the Extended Curie principle, is simple: Symmetries in an NK Boolean network (cause) generate symmetries in the emergent landscape (effect) such that basins of attraction occur in symmetry groups. We will argue that attractors are related to each other by the transforms that define symmetry groups and that the Boolean basin landscape is not a messy, random sort of thing; rather, it can be elegantly described in terms of symmetry groups and their relations.

At any given moment the non-chaotic Boolean system will and must deterministically fall into a single basin of attraction and cycle within that basin's attractor. The specific attractor the system ends up in depends on initial conditions and subsequent perturbations of the system. What is important is that, because attractors come in symmetry groups, information in the current attractor provides

information about other attractors in the same symmetry group. Thus local information has global implications in terms of movement across a landscape and a search for other attractors. It turns out, as we will describe below, that, within symmetry groups, these attractors are symmetry transforms of each other and, through these symmetry transforms, the attractors mutually imply each other. Mulvey, Amazeen, and Riley (2005, p. 308) note that symmetry relations among symmetry groups code information “that otherwise would need to be specified individually.” Quite simply, we propose that an attractor cycle in a Boolean landscape, through symmetry group relations, codes information about other possible attractor cycles. We have not yet defined symmetry groups in the Boolean context but will do so below. In this paper we shall begin to examine our proposal with simple simulations of landscape-generating processes that have a high degree of symmetry to establish the idea that attractors come in symmetry group in principle. But our ongoing work (Butner, Cooper, Malloy, & Smith, 2007) generalizes our findings for more complex cases. Let us now turn to elaborating symmetry groups in Boolean landscapes using the simple case of XOR-rings.

Start with Symmetry. Our initial strategy is to create systems with sufficient symmetry to allow an examination of the Extended Curie Principle; we start with Boolean XOR rings whose infrastructure (nodes, connections, logical operators) have a high degree of symmetry and whose effect (a Boolean landscape) should therefore have a high degree of symmetry. In this highly symmetrical context we can examine how the symmetries of the landscape are exhibited, how they are broken, and how they form symmetry groups. This work will be a basis for generalizing to Boolean systems that do not have this artificial degree of symmetry (Butner, Cooper, Malloy, & Smith, 2007). Boolean XOR Rings are a simple, mathematically and visually tractable case that can be used as a proof of concept. We will establish in XOR Ring case, first, that a landscape’s attractors emerge in symmetry groups

defined by symmetry transforms and, second, that these symmetry transforms are coded locally in the dynamics of each attractor in a symmetry group.

The Symmetry of Boolean XOR Rings

XOR Rings. An NK Boolean system is defined in terms of three critical variables, the number, N , of its nodes; the connections by which a node receives input from other nodes (and sometimes from itself); and the logical operators which determine if a node is ON or OFF iteration on $T+1$ (based on the states of its inputs at T).

Self-Referencing Nodes. In an NK Boolean system the connections among nodes refer to how the nodes take and give input. Obviously, K , the number of inputs per node is important but there are other aspects of the connections that will influence the symmetry of the attractor cycles that emerge when the system runs. Kauffman (1993) explored connecting nodes pseudo-randomly as did Malloy, Jensen and Song (2005) and Malloy, Bostic St Clair and Grinder (2005). But nodes can be connected any way. One potent variable is whether a node is connected to itself; that is, whether it examines its own state at time T (in relation to the states of other inputs at time T) to determine its state at time $T+1$. We call a node that examines its own state at time T a “self-referencing” node.

Rings. Figure 1 shows another aspect of the connections among nodes—the structure of the net within which the nodes reside. If the nodes are not connected pseudo-randomly then they must be connected in some systematic way. One way to connect nodes is to make a ring. In the case shown in Figure 4 the seven nodes take $K=2$ inputs each. They are all self-referencing nodes so one of the $K=2$ inputs is each node from itself; that leaves only one external connection by which a node receives input. In Figure 4, this single external input is a node's nearest clockwise neighbor. Thus the structure of the net is a ring.

XOR Operator. As detailed by Malloy and Jensen (2008) the XOR logical operator detects difference. By this we mean that in the ring structure above each self-referencing node will compare (at time T) its own state with its neighbors state; if its own state is the same as its neighbor's state (both ON or both OFF) then the node will be OFF at time $T+1$. Conversely if a node's own state at T is different than its neighbor's, it will be ON at $T+1$. Wolfram (2002, p. 25, rule 90) has established that the XOR operator produces an affine Sierpinski gasket in cellular automata, a result we will generalize to Boolean networks. The important point is that the Sierpinski gasket, like all fractals, is imbued with symmetry. For XOR Rings we will examine the XOR operator slightly differently than we have in the past when we used it as a basis for Boolean derivatives. Here our interest is in the process of emergence: Given that XOR is an operator that generates a known kind of symmetry (Sierpinski gasket) what happens to that symmetry in the Boolean landscape that emerges from a small system whose nodes all use XOR as their relational operator? We will look for symmetry and broken symmetry in the landscape of basins that emerges from such a system. Based on the work of Wolfram it is not hard to anticipate that the attractors will be related to affine Sierpinski fractals. Quite simply, the XOR operator makes a good starting point to explore the implications of the Extended Curie Principle as a principle of emergence.

Insert Figure 1 about here

Network Structure causes Landscape. Malloy, Bostic St Clair and Grinder (2005) argued that, computationally, the basin structure of Boolean system emerges from and is determined by the structural characteristics of the system. That is, the nodes, the connections among the nodes and the

logical operators that nodes use constitute one level of analysis. The basins of attraction and the attractors in each basin are the behavior of the system and, as such, constitute another level of analysis. In terms of the Curie principle, the structure of the system (nodes, connections, and operators) causes the landscape of attractor basins. This is analogous to the beads of water on the web emerging from the forces among the molecules in water. Thus we would expect to find symmetries in the effects (the basin landscape) when there is symmetry in the cause (XOR relations among nodes in a ring). Thus the highly symmetrical XOR Ring is expected to produce symmetry in the landscape of attractors. Most important, in terms of the Extended Curie Principle we expect the attractors to come in groups related by symmetry.

Assuming we can find symmetry groups in the basin landscape we will examine how a system might move along symmetry transforms from one basin of attraction to another. We are looking for formal descriptions, based on symmetry group theory, of how process might flow from one attractor to another.

Symmetry and Symmetry Breaking in the Basin Structure of an XOR Ring

Figure 2 shows the nine length-7 attractor cycles that emerge from the XOR relations and ring architecture of an N7 self-referencing XOR ring. To understand Figure 2 note that in a Boolean landscape attractor cycles are mathematically specified by $N \times L$ matrices, where N indicates the number of nodes and L indicates the length of the attractor cycle in iterations (see Malloy and Jensen, 2008; Malloy, Butner, Cooper, & Smith, 2007 for a more detailed motivation of this idea). In short, when a system with N nodes is in a cycle that repeats every L iterations, each node will repeat a pattern of ON, OFF states (0's and 1's in Boolean values) every L iterations. Thus, given N nodes, each one repeating a pattern every L iterations, the attractor cycle is completely characterized by an $N \times L$ matrix of 0's and

1's. This matrix, for purposes of visualization, can be converted to a grid of white and black cells (where the Boolean 0 = a white cell and 1 = a black cell). To be concrete, examine Figure 2 and look at the grid labeled A2, which simply means Attractor 2. The top row in the A2 grid corresponds to Node 1 (of the seven nodes shown in Figure 1) and Node 1's values are five WHITE cells followed by two BLACK cells. This means the Boolean values for Node 1 are {0000011} when the system is cycling in Attractor 2 (A2). Similarly, staying in the A2 grid, the row down is for Node 2, and its WHITE-BLACK PATTERN corresponds to the Boolean vector: {0000101}. Thus the A2 grid in Figure 2, taken across all seven rows, shows the pattern of the A2 attractor as all seven nodes cycle through seven iterations. It is obvious that an attractor cycle is a circle and has no natural starting point *per se*; thus by convention (Malloy, Jensen, & Song, 2005) for purposes of archiving matrices and for perceptually comparing the perceptual patterns generated by attractor matrices, we rotate each NxL attractor matrix to begin with the column vector that has the lowest binary value (reading down from the top). We call this convention of starting with the lowest valued column vector "normalization." Looking at the A3 grid in Figure 2 reveals that the seven nodes create a different perceptual pattern of black and white cells as they change across iterations than did A2; note that A3 has been normalized as are A4 through A10. In this way the attractor patterns, A2 to A10, shown in Figure 2 are visual representations of the Boolean dynamics of nine attractors in the landscape that emerges from the XOR Ring shown in Figure 1. This landscape has one other attractor, A1, that is not shown in Figure 2; A1 has a cycle length, L, equal to 1 and all seven nodes are OFF, that is they have a Boolean value of 0. If we were to visually represent A1 it would be a single column of WHITE cells (remember an attractor is characterized by an NxL matrix, thus for A1 where L=1 the matrix would be 7x1).

In Figure 2 some parts of an affine Sierpinski pattern may be apparent to the reader in some of

the attractor patterns. But neither the degree of symmetry nor how symmetry groups are defined is immediately obvious in Figure 2; the deeper forms of symmetry and of symmetry breaking and sharing require further analysis.

Insert Figure 2 about here

Tributaries and Stability. Basins are defined as an attractor and all its tributaries; in this paper we focus largely on attractors but tributaries will enter our analyses later. Basin 1, whose attractor is A1 and is not shown in Figure 2, has only one tributary; this tributary lasts one iteration during which all nodes are ON (Boolean value = 1) and would be represent visually as a seven-node column vector with all BLACK cells. All other basins (from 2 through 10) have seven tributaries leading into the attractor cycle. One way to think about attractor stability is through the consequences of a random perturbation to the system when the system is in a particular attractor. Attractors A2 to A10 each have seven state vectors that are in the attractor cycle itself; they also have seven state vectors that lead back into the attractor. If a random perturbation produces one of these fourteen state vectors, the system will remain in the same attractor, i.e., demonstrate stability in the face of that perturbation. In this way of thinking A2 to A10 are equally stable but A1 (whose cycle contains only one state vector and that has only one state vector as as tributary) is relatively unstable.

Derivatives and Phase Portraits: TAO and Meta-TAO analyses. Given that we can specify the attractor dynamics of a Boolean landscape with NxL attractor matrices (Malloy, Butner, Cooper, & Smith, 2007; Malloy & Jensen, 2008) we will now analyze a landscape's dynamics in two distinct ways, one way building on the other. First we find the discrete derivatives of the attractors, an analysis we call TAO (Malloy, Jensen, & Song, 2005). The TAO function, like derivatives, can be applied

recursively to its own output to generate higher order TAO levels (higher order derivatives). It will be convenient at times to refer the attractor cycle as TAO-0 and its derivatives as TAO-1, TAO-2, and so on. We will give a brief review of the TAO operator below; for now note that since TAO operates upon an NxL attractor matrix, its output is an NxL matrix that we call a TAO matrix. When TAO operates, recursively, on a TAO matrix (rather than an attractor matrix) it yields a higher order TAO matrix (higher order derivative).

A second analysis we call Meta-TAO; Meta-TAO is based on a general matrix operation that compares the Boolean values of any two conformable matrices, cell by cell, returning a 0 if the cells are the same and returning a 1 if the cells are different. In this paper our use of the Meta-TAO analysis will focus on Meta-TAO comparisons between an attractor matrix and its various TAO matrices (discrete derivatives). This use of Meta-TAO makes Meta-TAO conceptually related to the idea of a phase portrait. In classical oscillator theory (Abraham & Shaw, 1984) a phase portrait plots a value indicating the position of the oscillator against a value indicating its velocity (first derivative). Our Meta-TAO comparisons plot the momentary Boolean values of state vectors in an attractor cycle against derivative values indicating change in state vectors; this is not exactly the same as “position versus velocity” but, within the discrete case, Meta-TAO functions to reveal conceptual information about a system in the same way as does a phase portrait (Malloy, Butner, Cooper, Smith, & Dickerson, 2007). Given this overview, we will very briefly review the procedural details of these two analyses.

Calculating TAO. Figure 3a, top row, shows A2 and its first eight derivatives (TAO-1 through TAO-8). We will look at the logic of the TAO operation using a single node as an example. For the A2 matrix in Figure 3a note the top row (Node 1); as discussed above its Boolean values across the attractor cycle are {0000011}, where the vector is ordered from iteration 1 to iteration 7. This vector

shows that Node 1 does not change value across the first five iterations but, then, from the fifth to the sixth iteration it changes from 0 to 1; next from the sixth iteration (where its value is 1) to the seventh iteration (where its value is still 1) it does not change. Finally, from the seventh iteration back to the first iteration (it is a cycle), it changes from 1 to 0. We can capture the preceding verbal description in a vector of changes where 0 = no change and 1 = change. The pattern of such a change vector is {0000101}. Now, if you look at the first row (Node 1) of the TAO-1 matrix you will see the WHITE-BLACK cells correspond to the change vector {0000101}. Thus the TAO function returns a pattern of change in the Boolean values across time in the attractor matrix. If we expand across all nodes (rows), the TAO-1 matrix codes the changes in the Boolean values found in the A2 attractor matrix from one moment in time (iteration) to the next; thus it functions as a derivative. The same kind of relationship exists between TAO-1 and TAO-2, TAO-2 and TAO-3, etc.

Insert Figure 3 about here

Calculating Meta-TAO. Figure 3b, second row, shows eight Meta-TAO's (M1 through M8) of Attractor 2, each of which compares A2 (TAO-0) with one of its derivatives. As noted above, Meta-TAO does a Boolean comparison, cell by cell, of two conformable matrices; in Figure 3b all Meta-TAO's compare the A2 matrix with one of its derivatives. The first Meta-TAO (M0) in Figure 3b is all white because it is comparing the attractor (TAO-0) with itself; there are no differences, of course, between itself and itself. The M1 matrix compares the TAO-1 matrix in the top row with the A2 matrix in the top row. Notice that this comparison, which we call Meta-TAO-1, or more compactly M1, is exactly the same as the original basin. Note that in Figure 3 we have normalized the Meta-TAO matrices (rotated them to begin with the lowest valued column vector) so that they can be perceptually

compared with normalized attractor matrices. Thus if you calculate M1 by eye or by hand you will find that we have rotated M1 either 1 iteration to the left or, equivalently, 6 iterations to the right to obtain this identity of pattern between M1 and A2. The rationale for normalizing matrices was discussed above. Continuing this logic, the third matrix (M2) in the second row of Figure 3 is Meta-TAO-2; it compares (in the top row) the second derivative with the original attractor. M2, when it is rotated either 2 iterations to the left or five iterations to the right, is identical to the A2 matrix. For the moment we skip M3 and go on to M4, which, when rotated, is also identical to the original attractor.

Identity and Non-Identity Meta-TAO's. Meta-TAO's 1, 2, and 4 are identical, when rotated to normalized form, to the original attractor. Thus we can call them *identity* Meta-TAO's. In contrast, in Figure 3 we can see Meta-TAO's 3, 5, and 6 are different than the original attractor; a simple descriptive term for these is “non-identity” Meta-TAO's because whatever transformation, T , of TAO-0 these Meta-TAO's represent they certainly do not generate the identical TAO-0 matrix no matter how they are rotated. The conceptual lynch pin for the rest of this paper is determining the nature of T and determining how this transform relates to symmetry theory.

Meta-TAO-3. Now notice particularly Meta-TAO-3 (M3); it is not identical to A2 as were Meta-TAO-1 and Meta-TAO-2. Indeed, if you examine the nine attractors cycles presented either in Figure 2 or follow the arrow from M3 in Figure 3b to A4 in Figure 3c, you find that Meta-TAO-3 is identical to the attractor cycle for Basin 4 (A4) when both are rotated to normalized form. Even in the highly constructed and symmetrical case of an XOR Ring, this is provocative. We want to emphasize two things that happen when we use the Meta-TAO tool. First we have noted that certain Meta-TAO's (1, 2 and 4 in this case) are identity transforms: they produce the attractor cycle which is being analyzed. Second, certain Meta-TAO's (3 in this case) transform the current attractor cycle into a

different attractor in the same landscape. This second statement is important: Local analysis of local attractor dynamics yields the dynamic pattern of other attractors on the same landscape. Certain Meta-TAO matrices constitute a matrix transform, T , by which one attractor becomes another.

We will discuss below more fully the limitations inherent in this highly symmetrical XOR Ring case. For the moment one detail requires comment. The reader may have noticed in Figure 3 that TAO-3 and Meta-TAO-3 are identical. Thus the Meta-TAO-3 analysis was not required to define the transform, T , which changes the A2 matrix into the A4 matrix. Since TAO-3 and Meta-TAO-3 in this particular example are identical, T could have been defined by TAO-3, which is a prior and therefore simpler analysis. In our ongoing work (Butner, Malloy, Cooper, & Smith, 2007) with less symmetrical Boolean node architecture, including, even, randomly generated Boolean systems, the key conceptual points in this paper hold up. Moreover, when we use less symmetrical Boolean systems to generate landscapes than our current XOR Ring (Butner, Cooper, Malloy, T. E, & Smith, 2007) the Meta-TAO matrices will not be redundant with the TAO (derivative) matrices; rather it will be Meta-TAO matrices that act as the transform, T , to find other attractors. Thus, since we are using this simple case to build a methodology, we focus on the non-identity Meta-TAO's (such as Meta-TAO-3) here as the critical analyses for defining symmetry groups (below).

Emergence of symmetry groups on a Boolean landscape

We now have the requisite analyses for defining symmetry groups. Figure 4 shows crucial aspects of the symmetrical structure of the basin landscape of the N=7 XOR Ring; this system produces ten basins, each with a attractor cycle and some number of tributaries. Only the attractors (numbered 1 through 10) are shown in Figure 4. Tributaries are not shown. A complete representation of the full basin landscape would also show the tributaries but for our current conceptual point the structure of the

attractor cycle symmetry groups is sufficient. Each attractor cycle is shown for seven iterations (horizontal axis). As usual, the nodes are shown on the vertical axis. Note also that M5 and M6 both also lead to an attractor; these Meta-TAO's have the same properties as M3 and we could make the following arguments using either of them in place of M3.

Meta-TAO-3 as Symmetry Transform. Figure 4 summarizes symmetry relations among attractor cycles in the N7 XOR Ring based on the definition of the matrix operation:

$$T = [\text{Meta-TAO-3}].$$

Recall that Meta-TAO-3 compares TAO-0 (the original attractor cycle) with its third derivative (TAO-3). We review the methodology: If we take the TAO-0 matrix of 0's and 1's (shown as black and white cells in our figures) and lay the third derivative matrix over it, then Meta-TAO-3 will yield us a matrix of 0's and 1's where the 1's indicate that for a particular cell TAO-0 was different than TAO-3.

The circle of attractor cycles at the bottom of Figure 4 represents a symmetry group that results from from transforming each successive attractor matrix with its own Meta-TAO-3. Start with Attractor Cycle 2 at the top of the circle and operate on its NxL matrix with the Meta-TAO-3 matrix operation (**T**). **A2 T = A4**, where **T** is understood as operating on A2. Figure 3 already has shown the details of how operating on A2 with Meta-TAO-3 produces A4. Continuing this analysis, the Meta-TAO-3 transform applied to Attractor 4 is precisely Attractor 10, and so on, clockwise around the circle. Thus the Meta-TAO-3 transform defines an ordered set of attractors {2 ==> 4 ==> 10 ==> 5 ==> 9 ==> 3 ==> 6 ==> 2 ==> etc.}. Defining **T** as M3 generates a symmetry group because repeated applications of the transform remain in the group.

Insert Figure 4 about here

Now notice in Figure 4 that attractors 7 and 8 each have a circle leading back to themselves; this indicates that, for both of them, Meta-TAO-3 generates the original attractor. This is also true for A1. Note that A1 is a 7x1 all-WHITE column vector; that is, it is a fixed point attractor. A1 is shown at the top of Figure 7 as if repeating across 7 iterations (a 7x7 all-WHITE matrix) to make its representation parallel to the other attractors. These three attractors, A1, A7, A8, each comprise a symmetry group in which they are the only element—for each, the Meta-TAO-3 transform stays in a one-element group.

In the end we have four symmetry groups defined by the Meta-TAO-3 transform. A1, A7, and A8 each define a group with a single attractor. Attractors 2, 4, 10, 5, 9, 3, 6, in that order, define a fourth symmetry group. Mulvey, Amazeen, and Riley (2005) have noted that transforms that define symmetry groups are highly compressed codes for the large amounts of information required to specify the members of the groups individually. That is, if you have purely local information from Attractor 2 and you can perform the Meta-TAO-3 transform, you can derive any of the other attractors that are in the group with Attractor 2 without coding all the information required to specify the other attractor matrices individually. Besides this informational efficiency, these transforms—by their definition—are a way for a system, based purely on information that is *local* to a given attractor, to move among its attractor cycles, at least within symmetry groups. In terms of an adaptive walk on a landscape, transforms, such as Meta-TAO-3, which produce symmetry groups are alternative and efficient procedures for coding large parts of a landscape and, most importantly, they provide a formal and therefore well-defined basis for how to navigate from one attractor to other attractors based on information contained in a single attractor. There is a boundary condition on this principle. Notice that such transforms will not move a system between symmetry groups. In short, Meta-TAO-3 alone

will never take a system from Basin 2 to Basin 7.

A Symmetry-based method for “Fearless” Leaps in Adaptive Walks

As Kauffman (1993, p. 33) points out, adaptation usually proceeds through small, local changes. Such small perturbations are unlikely to move a system out of its current basin of attraction. Thus small local changes leave a system stuck in a local attractor. In contrast, large perturbations can propel a system into another basin but, in our metaphor, random, large perturbations are “fearful” leaps to unknown territory. As fearful, in Blake's metaphor, as the pervasive symmetry underlying natural process may be, here we have shown that symmetry can provide a new method for getting unstuck, for making, as it were, “fearless” leaps not into unknown territory but directly to other attractors via the transforms that define symmetry groups.

Symmetry as a Principle in Emergence

Goldstein (2002) proposes that emergence can be construed in terms of self-transcending constructions such as Cantor's Anti-diagonal Argument which formally incorporates, beyond simple recursion, novelty generating negation. We find Goldstein's (2002) self-transcending constructions and the negation that Goldstein argues is central to the emergence of radical novelty to have a tantalizing relationship to symmetry, particularly the negation implied by the breaking of symmetry. When a perfectly symmetrical sphere of milk is dropped into a bowl of milk the impact forms a symmetrical splash that resembles a crown with 24 spikes. It is always 24 spikes and this particular 24-pointed crown can be construed as emerging from the dynamic (and recursive) interactions of the liquid molecules. The crown is symmetrical but much less so than the sphere. There is an implicit negation in this process; the higher-order spherical symmetry is negated (broken) and a new, lower-order symmetry (the crown) appears. Does the crown transcend the structure of a sphere by breaking

the sphere's symmetry and bringing into being a new form? Moreover, as Stewart and Golubitsky point out, there are an infinite number of such 24-pointed crowns, each rotated some arbitrarily small number of degrees from all the others. Each time a sphere is dropped, a new crown (in the sense of rotated some arbitrarily small distance from any other crown that has ever occurred) appears; thus in a way each crown is radically new and will never be repeated. If we take the breaking of symmetry to be a kind of negation and if we assume that the complexity found in the form of the crown transcends the form of the sphere, then we have an argument parallel to Goldstein's and thus another candidate for a self-transcending construction in emergence. Well known principles of fluid dynamics underlie the breaking of the sphere into crown; and so the example is not a form of hylozoism based on symmetry as opposed to emergence (Goldstein, 2003). In the same way the appearance of symmetry groups in our Boolean landscape is a matter of derivations from the underlying nodes, connections and relational rules.

We now turn to examining the possibility that there is a lattice symmetry underlying the Meta-TAO symmetry groups we have found in this paper. Paralleling Goldstein's argument we propose the possibility that Boolean landscapes, or at least the ones studied in the paper, emerge through a process by which attractor patterns, arranged in symmetry groups, break out of a higher order symmetry.

A fractal kernel. We now expand our examination of the symmetry in the basin landscape of the N7 XOR Ring. The Meta-TAO matrix operation is quite general and can be applied to any two conformable matrices (attractors, TAO's, or Meta-TAO's). Figure 5 shows the result of using the Meta-TAO tool to compare two TAO-0 (attractor) matrices: A7 and A8. In short, if we lay Basin 7 and 8 attractor matrices over each other and XOR each cell, we find a pattern we call the “kernel,” which is shown in the middle of the Figure 5. The kernel can be rotated (one column to the right and one row

down) to produce the “rotated kernel.” We are using the term kernel in the fractal sense such as the buckled line that is the kernel of the Koch snowflake. This rotated kernel is in fact the attractor for Basin 2.

Insert Figure 5 about here

A Sierpinski Lattice. A lattice symmetry is one in which patterned tiles are translated horizontally and vertically and, in some cases, rotated (Stewart and Golubitsky, 1992, p. 51 and Chapter 4). Such tiling produces symmetrical patterns such as the Penrose fractal.

Given the definition of a symmetry lattice, imagine putting the attractor patterns from the {2, 4, 10, 5, 9, 3, 6} symmetry group (see Figure 4) on transparencies and then overlaying two of the attractor transparencies so that parts of the two patterns that are identical are placed behind each other. Next, add another attractor pattern behind the first two in a way that aligns those parts of the third pattern that are identical to the pattern seen through the first two transparencies. Continue adding attractor patterns, conforming each new pattern to the composite pattern showing through the previous transparencies. Doing so produces the affine Sierpinski lattice seen in Figure 6, in which the kernel, either rotated or unrotated, is easily found. Indeed the kernel can easily be seen as the tile that defines the Sierpinski Lattice.

The Conservation of Broken Symmetry. The assumption here is that the N7 XOR Ring constitutes a set of symmetrical generating process from which emerges a particular Boolean landscape and that the symmetry of the generating processes, via the Extended Curie Principle, is exhibited in the landscape in three ways. First, the N7 XOR Ring generates the symmetry of the Sierpinski Lattice shown in Figure 6. But this lattice symmetry is broken due to the constraints of L=7. How attractor

length affects the breaking of the Sierpinski symmetry is rational but beyond the scope of this paper. In Figure 6a the lattice symmetry is presented in its not-broken form so it is difficult to perceive the individual attractor patterns that break out of the lattice as the landscape emerges. So in Figure 6b, we have outlined the A6 attractor to make that attractor apparent. Indeed all of the attractors in the {2, 4, 10, 5, 9, 3, 6} symmetry group are contained in the Sierpinski lattice. The second way that the symmetry of the kernel tile breaks is into the pair of attractors, A7 and A8. The differences between those two attractors, cell by cell, reconstruct the Kernel Tile. The third symmetry is exhibited in A1 (an all-WHITE 7x1 column vector) which in terms of horizontal and vertical translation is perfectly symmetrical. While A1 is highly symmetrical, it is, as we have argued above, highly unstable and the dynamics of the system are not likely to stay in A1. It is, as it were, like the symmetry of a pencil balanced on its point (Stewart & Golubitsky, 1992, p. 12) waiting to fall. In terms of candidate principles for the concept of emergence there is something very intriguing about how, in this special case at least, the generating processes (a ring of XOR nodes) causes (in terms of the Extended Curie Principle) the Sierpinski symmetry of the XOR operator to be broken (negated) into fragments of the Sierpinski Lattice. Moreover, the breaking of symmetry occurs in groups defined by symmetry transforms such as Meta-TAO-3.

Insert Figure 6 about here

The Emergence of a Landscape from an N8 XOR Ring. We are exploring the idea, using Boolean landscapes as a case, that symmetry breaking may be deeply entwined with how emergence works. This idea is not fully complete in this paper and requires considerably more development since we are not explicating the details of how the number of nodes and attractor cycle length constrain the

landscape. But if we add one more case of an emergent Boolean landscape, the outlines of the argument will become more evident. We now describe how a slightly difference XOR Ring generates a profoundly different Boolean landscape. To begin this theoretical thread, consider briefly, the results of a simulation that uses an N8 XOR Ring as opposed to the results just shown for our N7 XOR Ring. Recall that N is simply the number of nodes in the ring; thus an N8 XOR Ring is identical to the N7 XOR Ring we have been discussing except that the ring has eight nodes. The N8 XOR Ring produces an extremely simple landscape with a single basin with a single attractor. The attractor has length $L=1$ and all the node values are 0 (OFF). It is a column vector of 0's. If we portray this attractor visually it would be single column of white cells. It is a 8×1 matrix that repeats endlessly unless perturbed.

Obviously, this single attractor of the N8 XOR Ring corresponds to the A1 attractor (a 7×1 all-WHITE column vector) of the N7 XOR Ring. For the N8 XOR Ring, the structure of the single basin on this landscape is such that all state (column) vectors in the state space eventually flow into this all-WHITE, fixed point attractor but they flow into it in an interesting way. First, there is a single column vector, all-BLACK, which is the only tributary that flows directly into the all-WHITE attractor vector. All other tributaries must flow first into the all-BLACK tributary and thence into the attractor. What do these other tributaries look like? There are 128 tributaries of length 7 iterations prior to the the all-BLACK vector; that is, these 128 tributaries are 7 column vectors long before they dump into the all-BLACK tributary. Put another way, these tributaries exist temporally for 7 iterations and then proceed into the all-BLACK tributary for one iteration before cycling endlessly in all-WHITE attractor. To help visualize the tributary structure, Figure 7 shows five examples from among the 128 length-seven tributaries; note that the first seven (counting from the left) state vectors in Figure 7 lead in every case to an all-BLACK column (the common tributary) and then to an all-WHITE vector (the attractor).

Notice in Figure 7 c (middle) that the pattern formed by the seven iterations leading up to the all-BLACK eighth column is similar in structure to the attractor pattern for Basin 10 (see either Figure 2 or 4) in the N7 XOR Ring while the N8 attractor in Figure 7a is similar to the N7 rotated kernel in Figure 5.

Insert Figure 7 about here

We have considered here XOR rings which are a very special case for at least two important reasons. First, the network structure, including connections among nodes and relational rules between nodes, is itself highly symmetrical. Since this network constitutes the generating processes from which the landscape emerges we might expect, via the Extended Curie principle (Stewart & Golubitsky, 1992) that the landscape too would exhibit symmetry. In fact, that was one hypothesis we wanted to test in the Boolean simulation context. Second, the XOR operator itself produces symmetry in the form of the Sierpinski gasket (Wolfram, 2002). The result, sensible but not inevitable, was that the perfect XOR ring generated a state space whose state vectors can be arranged into a Sierpinski Lattice (Figure 6) and this lattice represents a background symmetry available to be broken when the landscape self-organizes from XOR ring generating process.

Therefore, just like the Sierpinski lattice in Figure 6, all 128 length-eight tributaries of the N8 Ring could be arranged in a another Sierpinski lattice related to but scaled differently than the one shown in Figure 6. The vertical height of the N8 Sierpinski kernel would be eight nodes tall. Our proposal is that behind the N7 and the N8 systems are two related but slightly different Sierpinski lattices each of which breaks differently into various dynamical patterns as the state vectors of the two different systems flow from one iteration to the next across time. In the N7 Ring those dynamic flow

patterns stabilize as ten attractor cycles (the rest are tributaries) while in the N8 Ring only one state vector stabilizes as an attractor—the rest remain tributaries. How the N7 and N8 landscapes emerge in such different ways is beyond the scope of this paper. We can for now, using the outlines of Goldstein's argument, propose that symmetry breaking is a critical form of negation by which a radically new landscape emerges from the recursive interactions among Boolean nodes. Kaufman (2000) argues that algorithms by their nature cannot describe the radical novelty of the evolving universe. This is a critical examination of his own prior work and by extension of our work in this paper and, really, all mathematical and algorithmic models. To contextualize his point with our results, the breaking of symmetry in our small Boolean system hardly rises to the level of radical novelty. But if we construe the universe in general and life in particular as an open system imbued with symmetry what our results suggest is the negation of higher order symmetry as it breaks into more complex, lower order symmetry can be construed as a form of self-transcending construction generating the emergence of radically new forms in open systems. Bateson (2002, chapter 1) argues that life fundamentally is imbued with symmetry across and within species. He extends the symmetry argument to developmental processes taking the case of a frog's egg. In terms of breaking symmetry he argues (2002, p. 154) that in sexual reproduction epigenesis cannot begin until the spherical symmetry of an egg is broken along some completely unspecified (within the egg) meridian that will then define the bilateral symmetry of the frog. This breaking of symmetry is usually done by the sperm as it enters the egg but, as Bateson documents, can be successfully accomplished with a poke from a small hair. So beyond the genetic function of the sperm, it has a mechanical, symmetry breaking function. In the case of poking the egg with a hair, the frog will be perfectly formed but haploid and unable to reproduce.

If we assume that adaptive landscapes are useful models of open systems, our results provide a basis for thinking of symmetry and symmetry breaking as a frame of reference for how adaptation might work. In any event, the extended Curie principle operates well in the N7 and N8 XOR rings we have examined here: The symmetry of the node architecture is found, transformed, in a Boolean landscape.

Symmetry as a Principle in Evolution. In Boolean systems, the state vectors in the state space are bit strings that have been mapped, conceptually, onto the genotype for purposes of theorizing about adaptation and evolution. While we have not done so here it is typical in such discussions to assign a fitness potential to the wells (peaks) of Boolean landscape. These wells or peaks are the landscape's attractors. An adaptive walk moves via tributaries toward an attractor which is locally the most fit part of the landscape. We will continue to use the basin (rather than the peak) metaphor and therefore the potential well as an analogue to fitness.

Mutation (Holland, 1995, p. 76) functions through random or unsystematic alterations of one or more bits of information in the genotype bit string and such an alteration *could* transform the bit string (state vector) into a bit string that is a vector in a different basin and, subsequently, across time this vector would migrate to the attractor in that basin. When acting as the sole transformative agent, this chance-based approach, so important in neo-Darwinism, has been criticized (Kauffman, 1993, among others) as insufficient in the face of the massive order apparent in living systems; in this criticism there simply has not been enough time in the history of the universe to randomly perturb genomes and get all the extant complex genotypes, let alone for natural selection operate on each “experiment.” Within a simulation one benefit of random perturbation (mutation) is that it is possible to calculate the odds of a perturbation resulting in stabilizing the system into a different basins different from the current basin,

since, unlike the biological world, one can fully explore a Boolean system for every possible state. But, even in a simulation, mutation is based on an information minimalist approach which is contrary to the notion that systems are pattern rich. Still mutation may an important role in maintaining diversity when combined with crossover recombination.

Crossover recombination (e.g., Holland, 1995, p. 65, 69) is a simulation strategy that takes part of a genome string from one genotype (Parent 1) and the rest of the genome string from a second genotype (Parent 2) and combines them to produce a new genotype string (Child). In the Child's string of 0's and 1's, the point at which Parent 1's string ends the Parent 2's string begins is the crossover point. There can be more than one crossover point. Crossover recombination is generally taken as a model of biological reproduction, particularly biological crossover. In terms of self-transcending constructions, Goldstein (2004) proposes that Holland's crossover algorithm can be interpreted as negating the redundancies in recursion alone. Yet, in terms of fitness, these kinds of genetic algorithms tend to converge at local minima (in a basin model) or local maxima (in a peak model); consequently crossover recombination can leave a system with no way to leave a local optimization of fitness when, globally, other basins have attractors that are more fit. That is why it is useful computationally to add mutation to other genetic algorithms in a model since mutation adds random changes to the Child's string and thus prevents the population of genotypes (which are tending to converge to a local optimum) from becoming too similar; put another way, mutation increases diversity.

But something critical is missing from the mutation/crossover scenario. The adaptive leap, so made, may land, as it were, too close to the current adaptation or too far from from a new adaptation. Crossover converges on the same local optimum. In mutation most leaps, even if they make it to another basin, will be far from the attractor of the new basin. Mutation alone, crossover recombination

alone, and both together are not likely to land precisely on a different optimum. In the metaphor of this paper, adaptive leaps are fearful leaps.

The symmetry transforms found here suggest another possibility, the possibility of a fearless adaptive leap from one optimum directly to another. If there are underlying symmetries in the structure of a genome then this study suggests that the adaptive landscapes that emerge from the symmetries in the genetic structure will, by the Extended Curie Principle, be imbued with some degree to symmetry. And this symmetry, when broken (negated), can generate a novel landscape of attractors (optima) related by symmetry transforms. Following the logic of Mulvey, Amazeen, and Riley (2005) these symmetry groups code information about other possible attractor cycles in the same group. Thus these symmetry transforms provide the conceptual bases for fearless adaptive walks from one optimum to another.

Figures

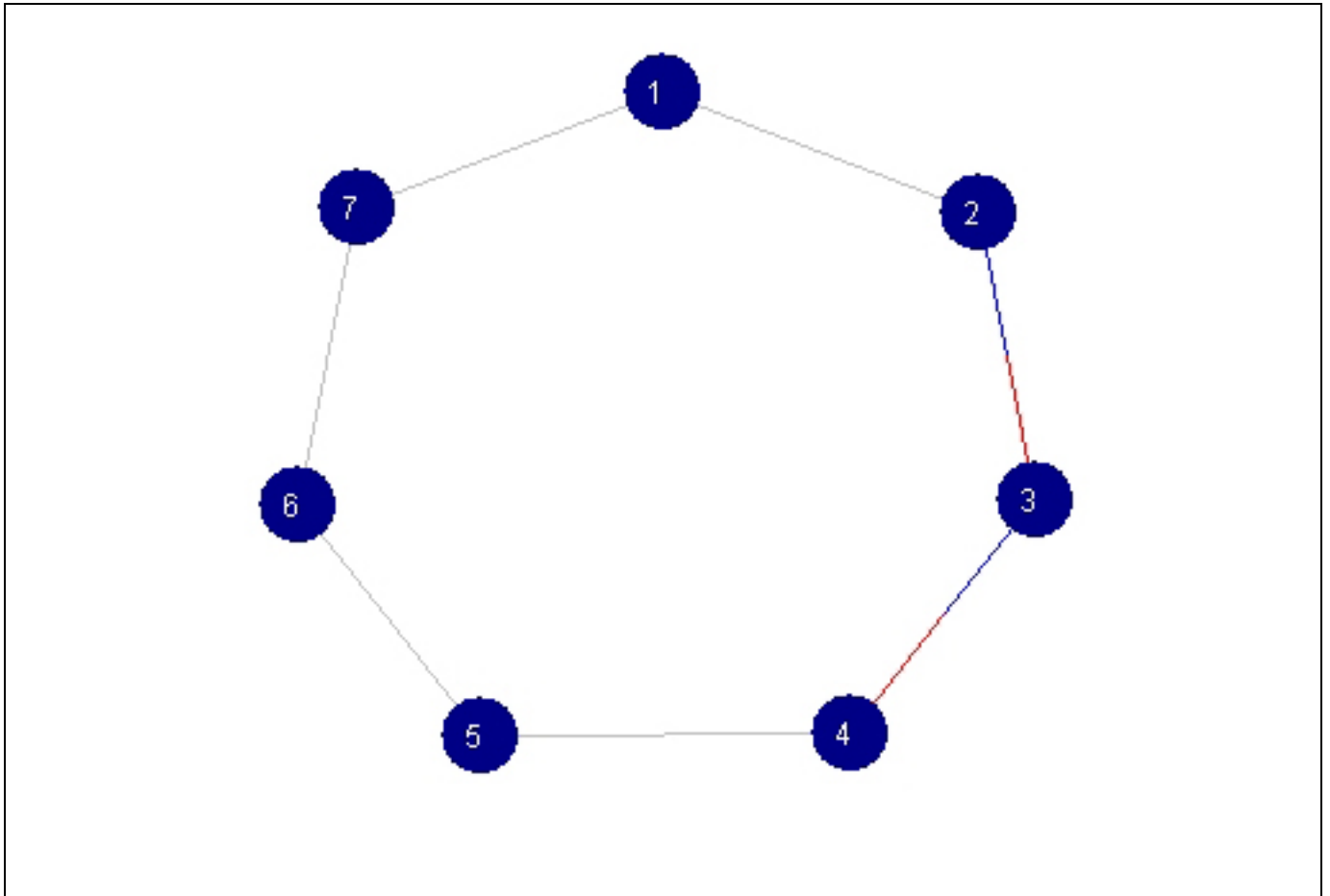


Figure 1. An $N=7$, $K=2$ Boolean XOR ring. Each node is self-referencing so one of its $K=2$ inputs is from itself; its other input is from its clockwise neighbor. The operator for each node is the XOR operator. The XOR operator determines that a node will be ON at time $T+1$ only if a node's state differs from its neighbor's state at time T .

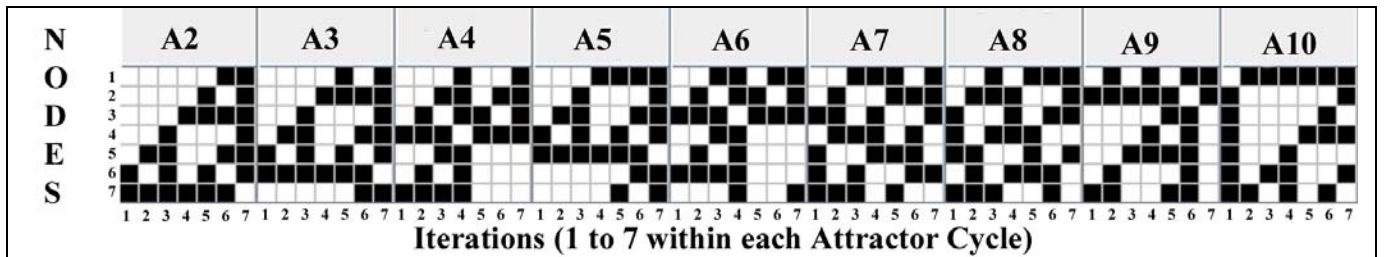
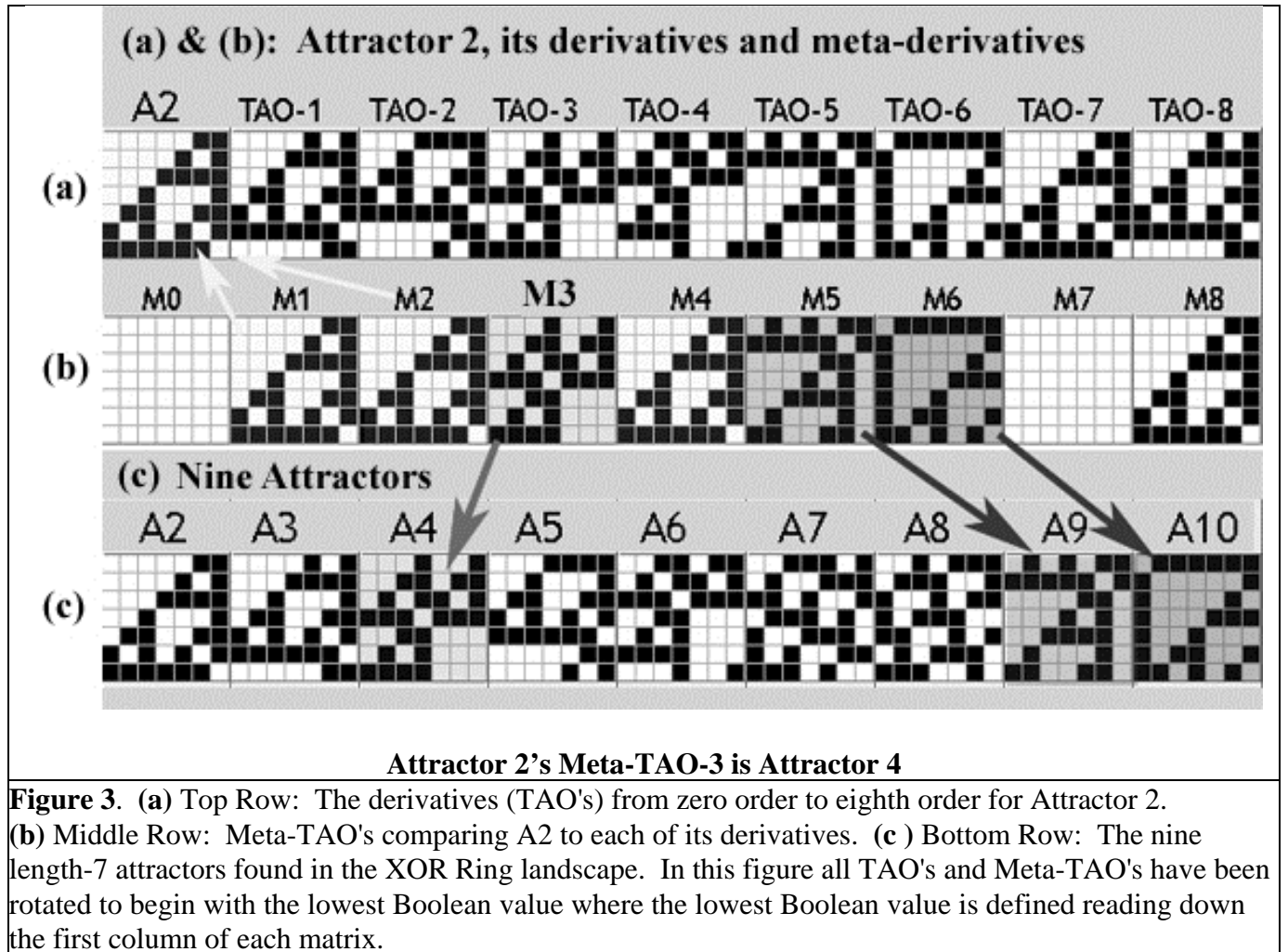


Figure 2. The landscape of the N7 Self-referencing XOR Ring has 10 basins each with an attractor. The attractor cycles of nine of the ten basins have length $L=7$ and are shown above, numbered A2 through A10. Iterations (1 to 7 fore each attractor) are on the horizontal axis; nodes are on the vertical axis. Basin 1 (not shown here) has an attractor of cycle length $L=1$ with all nodes OFF (WHITE).



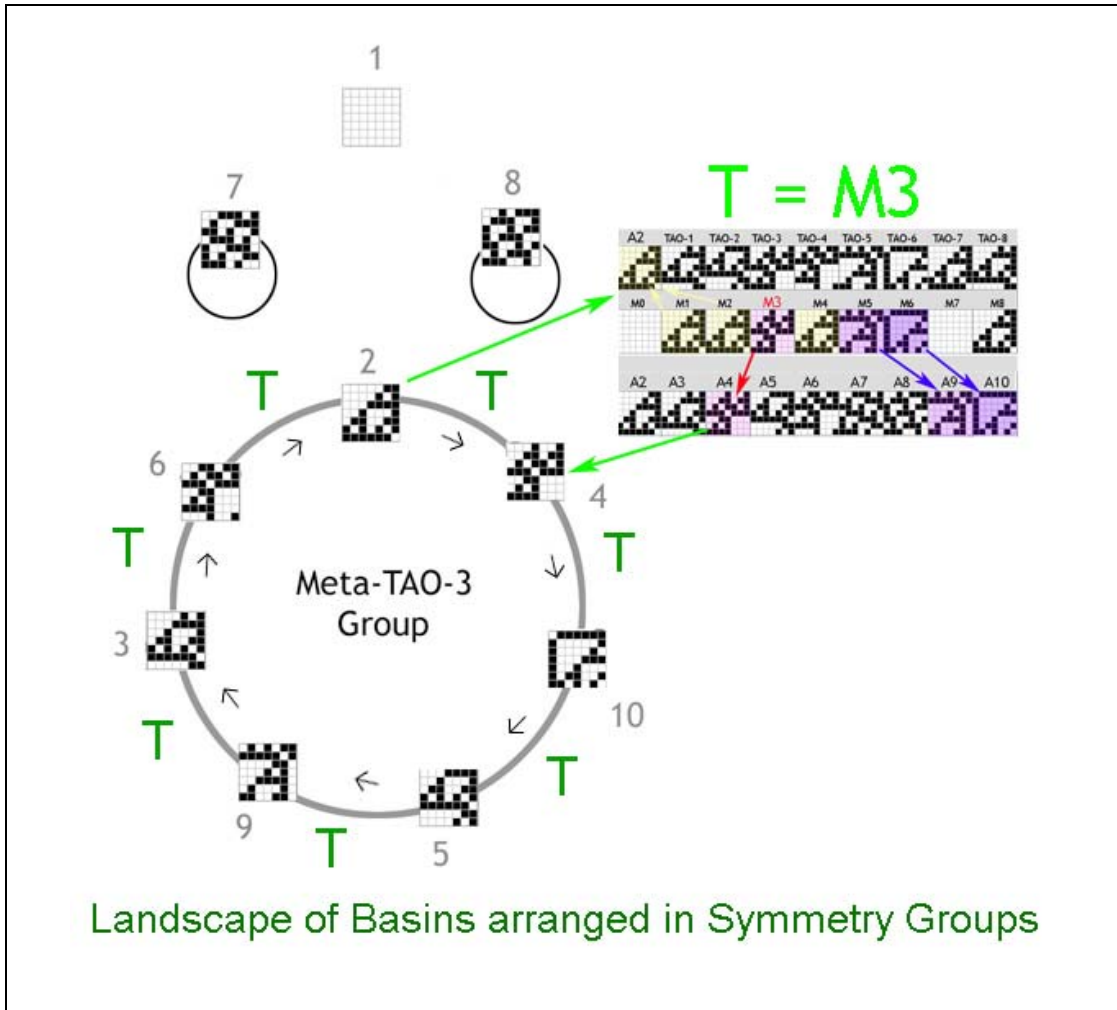


Figure 4. The basin landscape of the N=7 XOR Ring showing four symmetry groups. Only the attractors (which are numbered 1 through 10) are shown. Tributaries are not shown. Iterations are on the abscissa and nodes on the ordinate. Basin 1, though of L=1, is represented for 7 iterations to keep its image size comparable. All symmetry transforms, T, are Meta-TAO-3's of one Attractor leading into another attractor. Attractors A1, A7, and A8 each comprise a symmetry group in which they are the only element. Attractors A2, A4, A10, A5, A9, A3, and A6 comprise the fourth symmetry group.

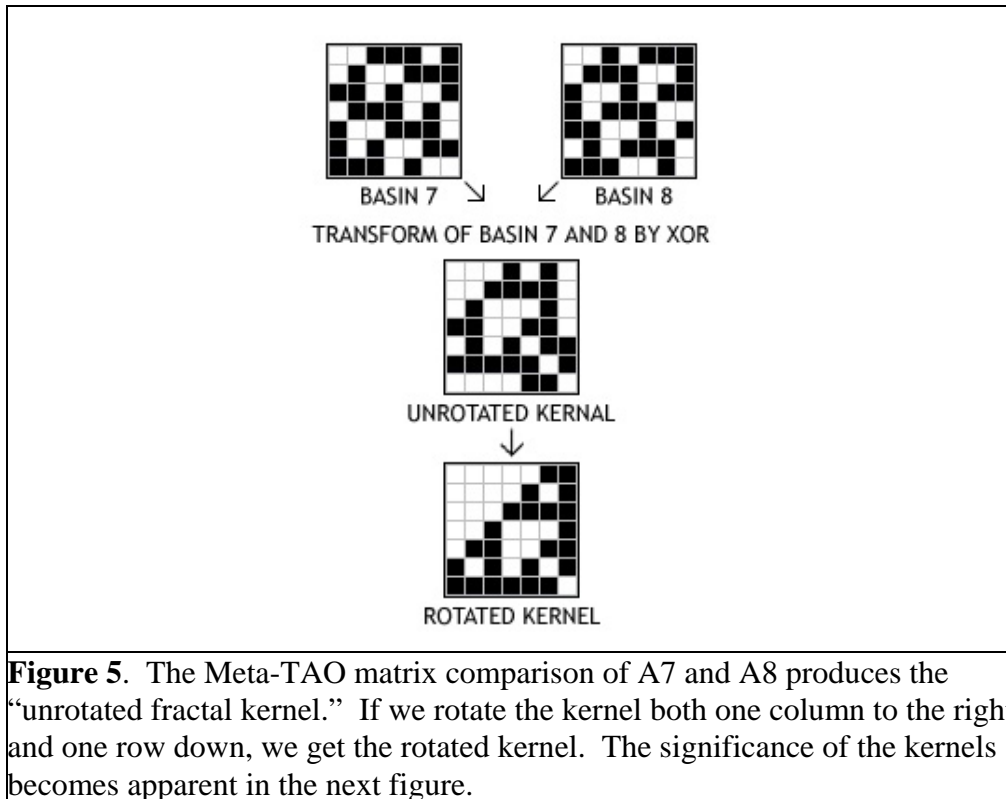


Figure 5. The Meta-TAO matrix comparison of A7 and A8 produces the “unrotated fractal kernel.” If we rotate the kernel both one column to the right and one row down, we get the rotated kernel. The significance of the kernels becomes apparent in the next figure.

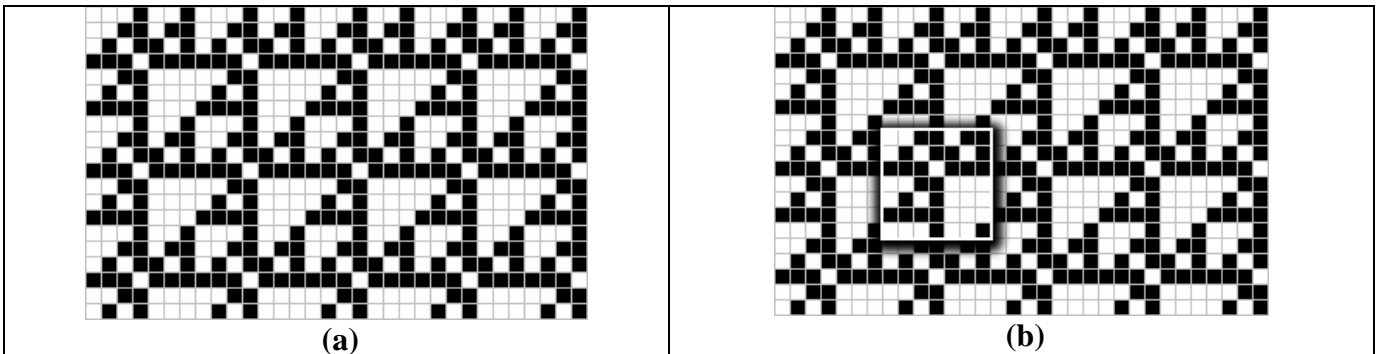
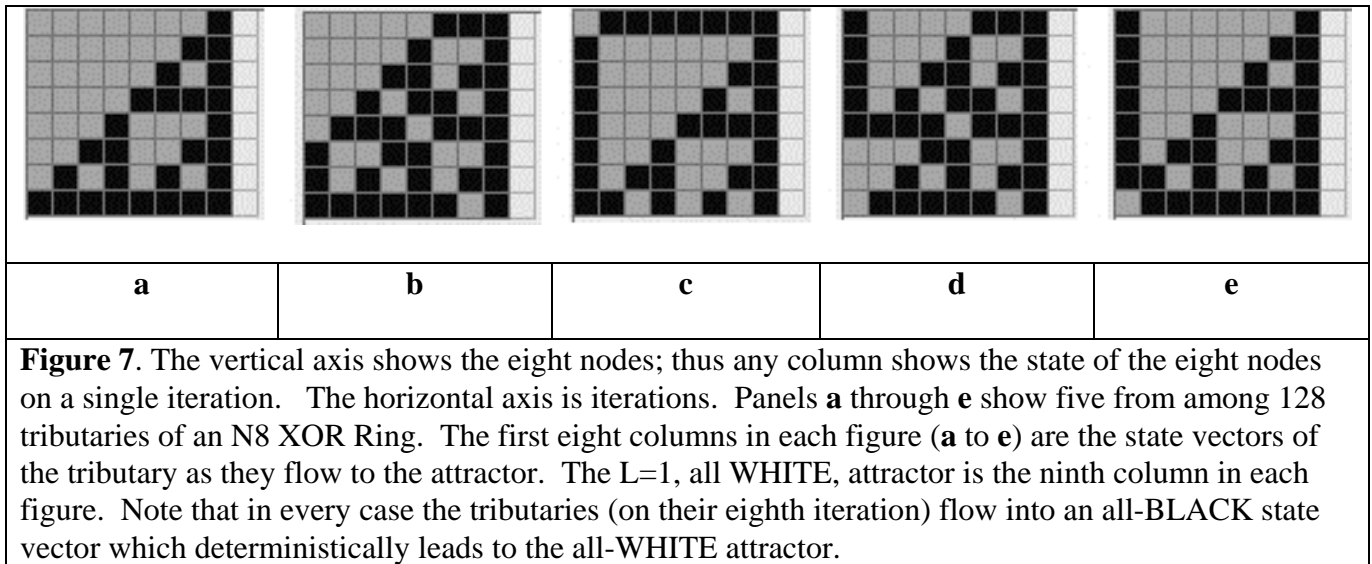


Figure 6. A Sierpinski lattice formed by overlaying all the attractors in the $\{2, 4, 10, 5, 9, 3, 6\}$ symmetry group. (a) If all the attractor patterns have been overlaid (as if on transparencies) so as to fit those parts of each with parts of other patterns that are identical, then, seen through all the transparencies, they form a lattice of small affine Sierpinski patterns. Note that the kernel, both rotated and unrotated can be easily found. (b) Any one of the attractors from the seven-member A2 through A6 symmetry group can be found by selecting the correct 7x7 square matrix from the lattice. Here, in (b), A6 is outlined as an example. The conjecture is that, when (potential) symmetry of the nodes of a the N7 XOR Ring breaks as the ring generates a Boolean landscape with attractors, the individual attractor patterns in the process of emerging from the dynamics of the Boolean net, break the higher order symmetry but as a group conserve (or share) the broken higher-order symmetry of the Sierpinski lattice.



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